A GRÜSS TYPE INEQUALITY FOR MAPPING OF BOUNDED VARIATION AND APPLICATIONS TO NUMERICAL ANALYSIS

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Abstract. A Grüss type inequality for Lipschitzian mappings and functions of bounded variation is given. Applications for the numerical integration problem of the Riemann-Stieltjes integral are also considered.

1. Introduction

In 1935, G. Grüss [9] proved an inequality which establishes a connection between the integral of a product of two functions and the product of the integrals. Namely, he has shown that:

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right|$$
$$\leq \frac{1}{4}(\Phi-\phi)(\Gamma-\gamma),$$

provided f and g are two integrable functions on $[a,b]$ and satisfy the condition:

$$\phi\leq f(x)\leq\Phi \text{ and } \gamma\leq g(x)\leq\Gamma$$

for all $x\in[a,b]$.

The constant $\frac{1}{4}$ is the best possible one and is achieved for

$$f(x)=g(x)=sgn\left(x-\frac{a+b}{2}\right).$$

In the recent paper [7], S.S. Dragomir and I. Fedotov proved the following results of Grüss type for Riemann-Stieltjes integral:

Theorem 1. *Let $f,u:[a,b]\to R$ be so that u is L-Lipschitzian on $[a,b]$, i.e.,*

$$|u(x)-u(y)|\leq L|x-y| \tag{1.1}$$

for all $x,y\in[a,b]$, f is Riemann integrable on $[a,b]$ and there exists the real numbers m,M so that

$$m\leq f(x)\leq M, \tag{1.2}$$

for all $x\in[a,b]$. Then we have the inequality

$$\left|\int_a^b f(x)du(x)-\frac{u(b)-u(a)}{b-a}\times\int_a^b f(t)dt\right|\leq\frac{1}{2}L(M-m)(b-a), \tag{1.3}$$

and the constant $\frac{1}{2}$ is sharp, in the sense it can not be replaced by a smaller one.

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For other results related to Grüss inequaity, generalizations in inner product spaces, for positive linear functionals, discrete versions, determinantal versions, etc., see Chapter X of the book [11] and the papers [1]-[10] , where further references are given.

In this paper we point out an inequality of Grüss type for Lipschitzian mappings and functions of bounded variation as well as its applications in numerical integration for the Riemann-Stieltjes integral.

2. Integral Inequalities

The following result of Grüss type holds.

Theorem 2. *Let $f, u : [a, b] \to R$ be such that u is L-lipschitzian on $[a, b]$, and f is a function of bounded variation on $[a, b]$. Denote by $\bigvee_a^b f$ the total variation of f on $[a, b]$. Then the following inequality holds*

$$\left| \int_a^b u(x)\, df(x) - \frac{f(b) - f(a)}{b - a} \times \int_a^b u(t)\, dt \right| \leq \frac{L}{2}(b - a) \bigvee_a^b f. \tag{2.1}$$

The constant $\frac{1}{2}$ is sharp, in the sense that it cannot be replaced by a smaller one.

Proof. As f is a function of bounded variation on $[a, b]$ and u is continuous on $[a, b]$, we have

$$\begin{aligned} & \left| \int_a^b u(x)\, df(x) - \frac{f(b) - f(a)}{b - a} \int_a^b u(t)\, dt \right| \\ = & \left| \int_a^b \left(u(x) - \frac{1}{b - a} \int_a^b u(t)\, dt \right) df(x) \right| \\ \leq & \sup_{x \in [a,b]} \left| u(x) - \frac{1}{b - a} \int_a^b u(t)\, dt \right| \bigvee_a^b f \end{aligned}$$

$$= \frac{1}{b - a} \sup_{x \in [a,b]} \left| \int_a^b [u(x) - u(t)]\, dt \right| \bigvee_a^b f. \tag{2.2}$$

Using the fact tha u is $L-$Lipschitzian on $[a, b]$, we can state, for any $x \in [a, b]$, that:

$$\begin{aligned} \left| \int_a^b [u(x) - u(t)]\, dt \right| & \leq \int_a^b |u(x) - u(t)|\, dt \leq L \int_a^b |x - t|\, dt \\ & = \frac{L}{2} \left[(x - a)^2 + (x - b)^2 \right] \end{aligned}$$

$$= L \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2} \right)^2}{(b - a)^2} \right] (b - a)^2 \leq \frac{L}{2} (b - a)^2, \tag{2.3}$$

and by (2.2)-(2.3) we get:

$$\sup_{x\in[a,b]}\left|u(x)-\frac{1}{b-a}\int_a^b u(t)\,dt\right|\leq\frac{L(b-a)}{2} \tag{2.4}$$

whence we obtain (2.1).
To prove the sharpness of the inequality (2.1), let us choose

$$u(x)=x-\frac{a+b}{2}\quad\text{and}\quad f(x)=\begin{cases}1 & \text{if}\quad x=a\\ 0 & \text{if}\quad a<x<b\\ 1 & \text{if}\quad x=b\end{cases}.$$

Then u is Lipschitzian with $L=1$, and f is of bounded variation. Also we have

$$\begin{aligned}&\int_a^b u(x)\,df(x)-\frac{f(b)-f(a)}{b-a}\times\int_a^b u(t)\,dt\\ =&\int_a^b\left(x-\frac{a+b}{2}\right)df(x)\\ =&\left(x-\frac{a+b}{2}\right)f(x)\Big|_a^b-\int_a^b f(x)\,dx=b-a.\end{aligned}$$

On the other hand, the right hand side of (2.1) is equal to $b-a$, and hence the sharpness of the constant is proved. ∎

The following corollaries hold:

Corollary 1. *Let $f:[a,b]\to R$ be as above and $u:[a,b]\to R$ be a differentiable mapping with a bounded derivative on $[a,b]$, that is, $\|u'\|_\infty=\sup_{t\in[a,b]}|u'(t)|<\infty$. Then we have the inequality:*

$$\left|\int_a^b u(x)\,df(x)-\frac{f(b)-f(a)}{b-a}\times\int_a^b u(t)\,dt\right|\leq\frac{\|u'\|_\infty}{2}(b-a)\bigvee_a^b f(x). \tag{2.5}$$

The inequality (2.5) *is sharp in the sense that the constant $\frac{1}{2}$ cannot be replaced by a smaller one.*

Corollary 2. *Let u be as above and $f:[a,b]\to R$ be a differentiable mapping whose derivative is integrable, i.e.,*

$$\|f'\|_1=\int_a^b|f'(t)|\,dt<\infty.$$

Then we have the inequality:

$$\left|\int_a^b u(x)f'(x)\,dx-\frac{f(b)-f(a)}{b-a}\times\int_a^b u(t)\,dt\right|\leq\frac{\|u'\|_\infty\|f'\|_1}{2}(b-a). \tag{2.6}$$

Remark 1. *If we assume that* $g:[a,b]\to\mathbb{R}$ *is continuous on* $[a,b]$ *and if we set* $f(x)=\int_a^x g(x)\,dx$, *then from* (2.6) *we get the following Grüss type inequality for the Riemann integral:*

$$\left|\frac{1}{b-a}\int_a^b u(x)\,g(x)\,dx-\frac{1}{b-a}\int_a^b u(t)\,dt\cdot\frac{1}{b-a}\int_a^b g(t)\,dt\right| \tag{2.7}$$
$$\leq \frac{\|u'\|_\infty\|g\|_1}{2}(b-a).$$

Corollary 3. *Assume that* $u:[a,b]\to\mathbb{R}$ *is differentiable on* (a,b), $u(a)\neq u(b)$, *and* $u':[a,b]\to\mathbb{R}$ *is bounded on* (a,b). *Then we have the trapezoid inequality:*

$$\left|\frac{u(a)+u(b)}{2}\cdot(b-a)-\int_a^b u(t)\,dt\right|\leq\frac{\|u'\|_\infty\|u'\|_1}{2[u(b)-u(a)]}\cdot(b-a)^2. \tag{2.8}$$

Proof. If we choose in Corollary 2, $f(x)=u(x)$, we get

$$\left|\int_a^b u(x)\,u'(x)\,dx-\frac{u(b)-u(a)}{b-a}\times\int_a^b u(t)\,dt\right|\leq\frac{\|u'\|_\infty\|u'\|_1}{2}\cdot(b-a) \tag{2.9}$$

Now (2.8) follows from (2.9). ∎

3. A Numerical Quadrature Formula for the Riemann-Stieltjes Integral

In what follows, we shall apply Theorem 2 to approximate the Riemann-Stieltjes integral $\int_a^b u(x)\,df(x)$ in terms of the Riemann integral $\int_a^b u(t)\,dt$.

Theorem 3. *Let* $f,u:[a,b]\to R$ *be as in Theorem 2 and*

$$I_h=\{a=x_0<x_1<...<x_{n-1}<x_n=b\}$$

a partition of $[a,b]$. *Denote* $h_i=x_{i+1}-x_i$, $i=0,1,...,n-1$. *Then we have:*

$$\int_a^b u(x)\,df(x)=A_n(u,f,I_h)+R_n(u,f,I_h) \tag{3.1}$$

where

$$A_n(u,f,I_h)=\sum_{i=0}^{n-1}\frac{f(x_{i+1})-f(x_i)}{h_i}\times\int_{x_i}^{x_{i+1}}u(t)\,dt \tag{3.2}$$

and the remainder term $R_n(u,f,I_h)$ *satisfies the estimation*

$$|R_n(u,f,I_h)|\leq\frac{L}{2}\nu(h)\bigvee_a^b f(x) \tag{3.3}$$

where $\nu(h)=\max_{i=0,\ldots,n-1}\{h_i\}$.

Proof. Applying Theorem 2 on the interval $[x_i, x_{i+1}]$, $i = 0, 1, ..., n-1$ we get

$$\left| \int_{x_i}^{x_{i+1}} u(x)\, df(x) - \frac{f(x_{i+1}) - f(x_i)}{h_i} \times \int_{x_i}^{x_{i+1}} u(t)\, dt \right| \leq \frac{L}{2} \cdot h_i \bigvee_{x_i}^{x_{i+1}} f(x).$$

Summing over i from 0 to $n-1$ and using the triangle inequality we obtain

$$\begin{aligned} \left| \int_a^b u(x)\, df(x) - A_n(u, f, I_h) \right| &\leq \frac{L}{2} \sum_{i=0}^{n-1} h_i \bigvee_{x_i}^{x_{i+1}} f(x) \leq \frac{L}{2} \cdot \nu(h) \sum_{i=0}^{n-1} \bigvee_{x_i}^{x_{i+1}} f(x) \\ &= \frac{L}{2} \cdot \nu(h) \bigvee_a^b f(x). \end{aligned}$$

and the corollary is proved. ∎

Remark 2. *Consider the equidistant partition I_h given by*

$$I_h : \ x_i = a + i \cdot \frac{b-a}{n}, \ i = 0, ..., n;$$

and define

$$\begin{aligned} & A_n(u, f, I_h) \\ = & \frac{n}{b-a} \sum_{i=0}^{n-1} \left[f\left(a + (i+1) \cdot \frac{b-a}{n}\right) - f\left(a + i \cdot \frac{b-a}{n}\right) \right] \times \int_{a+i \cdot \frac{b-a}{n}}^{a+(i+1) \cdot \frac{b-a}{n}} u(t)\, dt. \end{aligned}$$

Then we have

$$\int_a^b u(x)\, df(x) = A_n(u, f) + R_n(u, f)$$

where the remainder $R_n(u, f)$ satisfies the estimation

$$|R_n(u, f)| \leq \frac{L(b-a)}{2n} \cdot \bigvee_a^b f(x).$$

Thus, if we want to approximate the integral $\int_a^b u(x)\, df(x)$ by the sum $A_n(u, f, I_h)$ with an error of magnitude less than ε we need at least

$$n_0 = \left[\frac{L(b-a)}{2\varepsilon} \cdot \bigvee_a^b f(x) \right] + 1 \in \mathbb{N}$$

points.

Corollary 4. *Assume that u and f are as in Corollary 1. If I_h is as above, then* (3.1) *holds and the remainder term $R_n(u, f, I_h)$ satisfies the estimation*

$$|R_n(u, f, I_h)| \leq \frac{\|u'\|_\infty}{2} \cdot \nu(h) \bigvee_a^b f(x). \tag{3.4}$$

Corollary 5. *Assume that* u *and* f *are as in Corollary 2. Then* (3.1) *holds and the remainder term* $R_n(u,f,I_h)$ *satisfies the estimation*

$$|R_n(u,f,I_h)| \le \frac{\|u'\|_\infty \|f'\|_1}{2} \cdot \nu(h). \tag{3.5}$$

4. Application for the Trapezoidal Formula

The following integral inequality of trapezoid type for a mapping of bounded variation holds and has been proved by S. S. Dragomir in paper [8]. Here we give another proof based on the Grüss type inequality (2.1).

Theorem 4. *Let* $g:[a,b]\to R$ *be a mapping of bounded variation. Then we have the inequality:*

$$\left|\frac{g(a)+g(b)}{2}\cdot(b-a)-\int_a^b g(t)\,dt\right| \le \frac{b-a}{2}\cdot\bigvee_a^b g(t), \tag{4.1}$$

and the constant $\frac{1}{2}$ *is sharp.*

Proof. Using the integration by parts formula for the Riemann-Stieltjes integral we have:

$$\int_a^b \left(x-\frac{a+b}{2}\right)dg(x) = \frac{(b-a)(g(b)-g(a))}{2} - \int_a^b g(x)\,dx. \tag{4.2}$$

Define the mappings $f(x)=x-\frac{a+b}{2}$ and $u(x)=g(x)$. Then it is clear that f is L-lipschitzian with $L=1$ and u is of bounded variation. Applying the inequality (2.1), we deduce:

$$\begin{aligned}&\left|\int_a^b \left(x-\frac{a+b}{2}\right)dg(x) - \frac{g(b)-g(a)}{b-a}\cdot\int_a^b\left(x-\frac{a+b}{2}\right)dx\right| \\ \le\ & \frac{b-a}{2}\bigvee_a^b g(x).\end{aligned}$$

As $\int_a^b \left(x-\frac{a+b}{2}\right)dx=0$, then using the identity (4.2) we obtain the desired result (4.1).

Now consider the function $g:[a,b]\to\mathbb{N}$,

$$g(x) = \begin{cases} 1 & \text{if } x=a \\ 0 & \text{if } a<x<b \\ 1 & \text{if } x=b \end{cases}$$

Then g is of bounded variation and:

$$\int_a^b g(x)\,dx = 0, \quad \bigvee_a^b g(x)=2$$

Thus

$$\left|\frac{g(a)+g(b)}{2}\cdot(b-a)-\int_a^b g(t)\,dt\right|=b-a,$$

since $\frac{b-a}{2}\bigvee_a^b g(x)=b-a$, and the equality is realized in (4.1). The sharpness of the constant is thus proved. ∎

Corollary 6. *Let $g:[a,b]\to\mathbb{R}$ be a differentiable function whose derivative is integrable on (a,b). Then we have the inequality:*

$$\left|\frac{g(a)+g(b)}{2}(b-a)-\int_a^b g(t)\,dt\right|\leq\frac{b-a}{2}\|g'\|_1. \tag{4.3}$$

The following quadrature formula holds:

Theorem 5. *Let $g:[a,b]\to\mathbb{R}$ be a mapping of bounded variation and $I_h=\{a=x_0<x_1<...<x_{n-1}<x_n=b\}$ a partition of $[a,b]$. Denote $h_i=x_{i+1}-x_i$, $i=0,1,...n-1$. Then we have*

$$\int_a^b g(x)\,dx=T_n(g,I_h)+R_n(g,I_h) \tag{4.4}$$

where

$$T_n(g,I_h)=\sum_{i=0}^{n-1}\frac{g(x_{i+1})+g(x_i)}{2}\cdot h_i$$

and the remainder term $R_n(g,I_h)$ satisfies the estimate

$$|R_n(g,I_h)|\leq\frac{\nu(h)}{2}\cdot\bigvee_a^b g(x), \tag{4.5}$$

the constant $\frac{1}{2}$ being sharp.

Proof. Applying Theorem 4 on the interval $[x_i,x_{i+1}]$, we get

$$\left|\frac{g(x_{i+1})+g(x_i)}{2}\cdot h_i-\int_{x_i}^{x_{ii+1}} g(t)\,dt\right|\leq\frac{h_i}{2}\bigvee_{x_i}^{x_{i+1}} g(x)$$

for all $i=0,1,...,n-1$.
Summing over i from 0 to $n-1$, we deduce

$$\begin{aligned}\left|\int_a^b g(x)\,dx-T_n(g,I_h)\right| &\leq \frac{1}{2}\sum_{i=0}^{n-1}\bigvee_{x_i}^{x_{i+1}} g(x)\,h_i\\ &\leq \frac{1}{2}\nu(h)\sum_{i=0}^{n-1}\bigvee_{x_i}^{x_{i+1}} g(x)=\frac{1}{2}\nu(h)\bigvee_a^b g(x).\end{aligned}$$

The case of the equality follows from Theorem 5 and we omit the details. ∎

Remark 3. *Consider the equidistant partition I_h given by*

$$I_h : x_i = a + i \cdot \frac{b-a}{n},\ i = 0, ..., n;$$

and define

$$T_n(g) = \frac{b-a}{2n} \sum_{i=0}^{n-1} \left[g\left(a + (i+1) \cdot \frac{b-a}{n} \right) + g\left(a + i \cdot \frac{b-a}{n} \right) \right]$$

Then we have:

$$\int_a^b g(x)\, dx = T_n(g) + R_n(g)$$

where the remainder $R_n(g)$ satisfies the estimation

$$|R_n(g)| \leq \frac{(b-a)}{2n} \bigvee_a^b g(x).$$

Thus, if we want to approximate the integral $\int_a^b g(x)\, dx$ by the sum $T_n(g)$ with an error of magnitude less than ε we need at least

$$n_0 = \left[\frac{(b-a)}{2\varepsilon} \bigvee_a^b g(x) \right] + 1 \in N$$

points.

Corollary 7. *Let $g : [a,b] \to R$ be a differentiable function whose derivative is integrable on (a,b). If I_h is as above, then (4.4) holds and the remainder satisfies the estimation:*

$$|R_n(g, I_h)| \leq \frac{\nu(h)}{2} \|g'\|_1.$$

5. Quadrature Methods for the Riemann-Stieltjes Integral of Continuous Mappings

Let $I_h = \{a = x_0 < x_1 < ... < x_{n-1} < x_n = b\}$ be a partition of $[a,b]$ and denote $h_i = x_{i+1} - x_i$, $i = 0, 1, ...n-1$.

We start to the following lemma which is of interest in itself.

Lemma 1. *Let f be a function from $C[x_i, x_{i+1}]$, i.e., f is continuous on $[x_i, x_{i+1}]$, and let u be a function of bounded variation on the same interval. The following inequality holds:*

$$\left| \int_{x_i}^{x_{i+1}} f(x)\, du(x) - [u(x_{i+1}) - u(x_i)] \bar{f}_i \right| \leq \omega[f, h_i] \bigvee_{x_i}^{x_{i+1}} u(x) \tag{5.1}$$

where $\bar{f}_i = \dfrac{1}{h_i} \displaystyle\int_{x_i}^{x_{i+1}} f(x)\, dx$, $h_i = x_{i+1} - x_i$, $\omega[f, h_i] = \sup\limits_{|x-t| \leq \delta} |f(x) - f(t)|$, is the modulus of continuity.

Proof. Since $f \in C[x_i, x_{i+1}]$ and u is a function of bounded variation, by the well known property of such couple of functions we have

$$\left| \int_{x_i}^{x_{i+1}} f(x)\, du(x) \right| \le \|f\|_{C[x_i, x_{i+1}]} \bigvee_{x_i}^{x_{i+1}} u(x).$$

Therefore,

$$\left| \int_{x_i}^{x_{i+1}} f(x)\, du(x) - [u(x_{i+1}) - u(x_i)]\, \bar{f}_i \right| =$$

$$\begin{aligned}
&= \left| \int_{x_i}^{x_{i+1}} \left[f(x) - \bar{f}_i \right] du(x) \right| \\
&\le \left\| f(x) - \bar{f}_i \right\|_{C[x_i, x_{i+1}]} \bigvee_{x_i}^{x_{i+1}} u(x) \\
&= \sup_{x \in [x_i, x_{i+1}]} \left| f(x) - \bar{f}_i \right| \bigvee_{x_i}^{x_{i+1}} u(x) \\
&= \frac{1}{h_i} \sup_{x \in [x_i, x_{i+1}]} \left| h_i f(x) - h_i \bar{f}_i \right| \bigvee_{x_i}^{x_{i+1}} u(x) \\
&= \frac{1}{h_i} \sup_{x \in [x_i, x_{i+1}]} \left| \int_{x_i}^{x_{i+1}} f(x)\, dt - \int_{x_i}^{x_{i+1}} f(t)\, dt \right| \bigvee_{x_i}^{x_{i+1}} u(x) \\
&= \frac{1}{h_i} \sup_{x \in [x_i, x_{i+1}]} \left| \int_{x_i}^{x_{i+1}} [f(x) - f(t)]\, dt \right| \bigvee_{x_i}^{x_{i+1}} u(x) \\
&\le \sup_{x \in [x_i, x_{i+1}], t \in [x_i, x_{i+1}]} |f(x) - f(t)| \bigvee_{x_i}^{x_{i+1}} u(x) \\
&= \omega[f, h_i] \bigvee_{x_i}^{x_{i+1}} u(x),
\end{aligned}$$

and the lemma is proved. ∎

Let $f \in C[a, b]$. The dual to $C[a, b]$ is the space of functions of bounded variation, the general form of the functional on $C[a, b]$ is $I(f) = \int_a^b f(x)\, du(x)$ where u belongs to the space of functions of bounded variation.

That is why in the theory of quadrature methods for continuous functions the case of the integrals of such a type is the most interesting. For the integral with continuous integrand $\int_a^b f(x)\, w(x)\, dx$ $(w(x) > 0)$ we introduce the error functional for the quadrature rule (with the weights s_{nk} and nodes x_k)

$$I(f) \equiv \int_a^b f(t)\,w(t)\,dt \approx -\sum_{k=0}^{n} f(x_k)\,s_{nk} \equiv I_n(f)$$

by the formula

$$E_n(f) = I(f) - I_n(f).$$

Here is the more general result for the composite quadrature rules for the functions from $C[a,b]$.

Theorem 6. *Let a and b be finite real numbers and let $f,u:[a,b]\to R$ be such that $f\in C[a,b]$ and u is a function of bounded variation on $[a,b]$. If $a=x_0<x_1<...<x_{n-1}<x_n=b$ is a division of $[a,b]$ such that $|h_i|<\delta, \forall i=0,1,...,n-1$ where $h_i=x_{i+1}-x_i$, then the following estimation for the Error functional of the Riemann-Stieltjes quadrature rule is true*

$$\left|E_n^{comp}(f)\right| \le \omega[f,\delta]\cdot\bigvee_a^b u(x), \tag{5.2}$$

where

$$E_n^{comp}(f) = \int_a^b f(x)\,du(x) - \sum_{i=0}^{n-1}\frac{u(x_{i+1})-u(x_i)}{h_i}\times\int_{x_i}^{x_{i+1}} f(x)\,dx,$$

and $\omega[f,\delta]$ is the modulus of continuity of f with respect to δ.

Proof. For a given division of $[a,b]$ as above, we have

$$\int_a^b f(x)\,du(x) = \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} f(x)\,du(x).$$

Then by Lemma 1, we can write successively:

$$\begin{aligned}
\left|E_n^{comp}(f)\right| &= \left|\int_a^b f(x)\,du(x) - \sum_{i=0}^{n-1}\frac{u(x_{i+1})-u(x_i)}{h_i}\times\int_{x_i}^{x_{i+1}} f(x)\,dx\right| \\
&= \left|\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} f(x)\,du(x) - \sum_{i=0}^{n-1}\frac{u(x_{i+1})-u(x_i)}{h_i}\times\int_{x_i}^{x_{i+1}} f(x)\,dx\right| \\
&\le \sum_{i=0}^{n-1}\left|\int_{x_i}^{x_{i+1}} f(x)\,du(x) - \frac{u(x_{i+1})-u(x_i)}{h_i}\int_{x_i}^{x_{i+1}} f(x)\,dx\right| \le \\
&\le \sum_{i=0}^{n-1}\omega[f,h_i]\bigvee_{x_i}^{x_{i+1}} u(x) \le \omega[f,\delta]\sum_{i=0}^{n-1}\bigvee_{x_i}^{x_{i+1}} u(x) \\
&= \omega[f,\delta]\bigvee_a^b u(x),
\end{aligned}$$

and the theorem is proved. ∎

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